Source Localization Using TDOA with Sensor Position Errors Based on Constrained Total Least Squares and ADMM

Wen Ma, Hongyan Zhu

School of Electronic & Information Engineering

Xi'an Jiaotong University

Xi'an, China 710049

hyzhu@mail.xjtu.edu.cn

Abstract— **Source localization for the nonlinear measurement model based on time difference of arrival (TDOA) measurements remains a vital research area and has been intensively studied for the past few decades. However, the localization accuracy decreases significantly as the random measurement noise becomes large. In addition, when sensors are mounted on moving platforms like vehicles or aircrafts, inevitable sensor position errors might pose more severe challenges on source localization accuracy. This paper proposes to construct a pseudo-linear measurement model that introduces both the TDOA measurement noise and the sensor position error firstly. Next, the constrained total least squares (CTLS) formulation is presented, and the iterative alternating direction method of multipliers (ADMM) is employed to solve the resulting optimization model. Simulation results show that the proposed method can approach Cramer Rao lower bound (CRLB) better and outperforms several existing methods when considering sensor position uncertainties and large TDOA measurement errors.**

Keywords—Source localization, time difference of arrival, alternating direction methods of multipliers, constrained total least squares.

I. Introduction

In recent years, the problem of passive source localization by a sensor network has drawn considerable attention in the signal processing community owing to its importance in numerous applications including wireless communications, surveillance, navigation, and geophysics [1]-[3].

Various passive localization methods [4]-[9] have been proposed by using different types of sensor measurements, including angle of arrival (AOA), time of arrival (TOA), time difference of arrival (TDOA), frequency difference of arrival (FDOA), received signal strength (RSS) and a hybrid of these. Compared with the localization method using TOA, the TDOA-based source localization has the advantage that there is no need to synchronize sensor clocks with that of the target [10].

The TDOA measurement is evaluated by calculating the differences in arrival times of the source signal at multiple pairs of sensors. Each TDOA measurement defines a hyperbolic on which the source must lie, and the position estimate is given by the intersection of two or more hyperbolas in ideal conditions. However, the nonlinear hyperbolic equations become inconsistent as the TDOA measurements are corrupted by noises, and the hyperbolae no longer intersect at a single point. Finding the solution of the hyperbolic equation is generally not a trivial task due to its nonlinear and nonconvex nature.

To handle the nonlinearity in the least-squares (LS) framework, one method is to conduct the local linear approximation by the Taylor-series expansion [11], which requires a proper initial guess that are close to the true solution to avoid local optima. Such a technique can achieve a maximum likelihood (ML) approximate solution under the Gaussian measurement noise assumption, but the global optima is not guaranteed to be obtained. [12] relaxed the nonconvex ML optimization to a convex optimization problem via the semidefinite programming. Another alternative solution is given in closed form [13]-[14], which reconstructed a pseudo-linear LS by introducing an intermediate variable. Chan [14] applied two-step weighted least-squares (TSWLS) to obtain more accurate source position using the relationship between the source position and the auxiliary variable, and the corresponding Cramer-Rao lower bound (CRLB) has been derived. In addition, many improved versions based on the pseudo-linear LS have also been proposed, including the constrained weighted least-squares (CWLS) [15], bias reduction methods [16], the convex relaxation methods [6], and many others [17]-[19].

The methods mentioned above are with respect to a set of sensors with exactly known positions, but accurate sensor positions are not always available in practice [20-22]. There have been several fundamental problems and corresponding approaches with sensor position uncertainties, including node localization of wireless sensor networks (WSNs) [12], sensor position calibration [23], and target localization in the presence of sensor position errors (SPE) [24]-[26]. The first two kinds of problems focus on obtaining more accurate sensor positions, but the solutions still have estimation errors. The third kind addressed the source localization problem in the presence of random small SPE. As shown in [25], source localization accuracy is extremely sensitive to SPE, which is reflected by the fact that even small change in SPE can lead to a significant decrease in the estimation performance of source position, especially under poor location geometry and far-field source scenarios. In [24]-[25], source localization problem in the presence of SPE was constructed in the framework of linear LS. Similar to [13]-[14], the solution was performed by the two-step WLS method in closed form or convex relaxation methods for CWLS [26], where the update of weights requires the information related to both the measurement noise and SPE. In addition, the total least square (TLS) method can achieve a better accuracy than the ordinary LS when the measurement data are perturbed by noise [29]- [31]. The constrained total least-squares (CTLS) method [29] is another well-known iterative approach. Based on the CTLS technique, an iterative technique in line with Newton's method is developed to provide efficient numerical solutions with a lower computational cost. However, it may have problems of non-convergence. When the measurement error is large, the error caused by the squared term also increases, leading to a reduction in the final localization accuracy.

The alternating direction method of multipliers (ADMM) [32] is commonly used to solve optimization problems with only equational constraints in the presence of two (or more) optimization variables. The ADMM method performs separable minimization over different primal variables in successive steps. Many numerical experiments show that the ADMM has a slower convergence rate, especially near the optimal solution [31]. To improve the convergence speed, the ADMM algorithm can be combined with an optimization algorithm, such as Newton's method [33]. In particular, the convergence of the ADMM is explored by Eckstein and Bertsekas in [34]. To improve the localization performance in the presence of SPE and large TDOA measurement errors, we establish an optimization model in the form of CTLS and solve it by the ADMM scheme.

Main contributions of this work include:

- taking into account both sensor position errors and random TDOA measurement errors, construct a pseudo-linear measurement equation.
- establish the CTLS optimization model and develop the solution scheme by using ADMM.
- verify the superior performance of the proposed method with smaller estimation bias and RMSE.

The rest of this paper is organized as follows: The TDOA-based measurement model and our assumptions are described in Section II. The optimization problem in the form of CTLS is established in Section III-A. In Section III-B, the proposed ADMM solution to the established CTLS model is detailed. Simulation results are presented in Section IV to evaluate the performance of the proposed algorithm and the conclusions are presented in Section V.

II. TDOA-Based Localization Model

Consider a passive localization system consisting of one target and M sensors $\mathbf{S} = \{\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_M\}$ in a two dimensional (2D) right-handed rectangular coordinate system. It is firstly assumed that the positions of the observation stations are known exactly and the coordinates of the ith sensor are represented by $\mathbf{s}_i = (s_{ix}, s_{iy})^T, i = 1, 2, \cdots, M$, where $(\cdot)^T$ is the transpose operation. The unknown target location is denoted by $\mathbf{x} = (x, y)^T$. It is assumed that the first sensor is the reference sensor and r_i is the distance from the source to sensor i. Based on the TDOA measurements, we can obtain the basic measurement model as follows

$$\begin{aligned} r_{i1} &= r_i - r_1 \\ &= \sqrt{(x - s_{ix})^2 + (y - s_{iy})^2} - \sqrt{(x - s_{1x})^2 + (y - s_{1y})^2}, \\ i &= 2, 3, \cdots, M, \end{aligned} \tag{1}$$

where r_{i1} denotes the range difference of arrival between distance r_i and r_1.

Finding the solution of (1) is generally not a trivial task due to its non-linear nature. To overcome the difficulty, by squaring both sides of (1) and introducing the auxiliary variable, we have

$$\begin{aligned} &(s_{ix} - s_{1x})(x - s_{1x}) + (s_{iy} - s_{1y})(y - s_{1y}) + r_{i1} r \\ &= \frac{1}{2}\left((s_{ix} - s_{1x})^2 + (s_{iy} - s_{1y})^2 - r_{i1}^2\right). \end{aligned} \tag{2}$$

where $r \triangleq r_1$ is the auxiliary variable. Collecting all TDOA measurements, (2) can be formulated in the following matrix

$$\mathbf{A}\boldsymbol{\theta} = \mathbf{b}, \tag{3}$$

where

$$\mathbf{A} = \begin{bmatrix} s_{2x} - s_{1x} & s_{2y} - s_{1y} & r_{21} \\ s_{3x} - s_{1x} & s_{3y} - s_{1y} & r_{31} \\ \vdots & \vdots & \vdots \\ s_{Mx} - s_{1x} & s_{My} - s_{1y} & r_{M1} \end{bmatrix}, \boldsymbol{\theta} = \begin{bmatrix} x - s_{1x} \\ y - s_{1y} \\ r \end{bmatrix},$$

$$\mathbf{b} = \frac{1}{2}\begin{bmatrix} (s_{2x} - s_{1x})^2 + (s_{2y} - s_{1y})^2 - r_{21}^2 \\ (s_{3x} - s_{1x})^2 + (s_{3y} - s_{1y})^2 - r_{31}^2 \\ \vdots \\ (s_{Mx} - s_{1x})^2 + (s_{My} - s_{1y})^2 - r_{M1}^2 \end{bmatrix}.$$

Now, we remove the assumption that sensor positions are known exactly [23], [24], and instead model the available sensor position by

$$\mathbf{s}_i = \mathbf{s}_i^0 + \Delta\mathbf{s}_i, \; i = 2, 3, \ldots, M. \tag{4}$$

where $\mathbf{s}_i$, $\mathbf{s}_i^0$ and $\Delta\mathbf{s}_i$ represent the nominal sensor position, true position and sensor position errors, respectively. The reference sensor is typically an anchor node, and we assume that its position is known precisely. In addition, considering the measurement noise of TDOA, the measurement model becomes

$$r_{i1} = r_i^0 - r_1^0 + n_{i1}, \; i = 2, 3, \ldots, M. \tag{5}$$

The matrix form is

$$\mathbf{A}\boldsymbol{\theta} - \mathbf{b} = \varepsilon \tag{6}$$

where n_{i1} and ε are corresponding noise terms.

III. Proposed Source Localization Method

For the source localization problem using TDOA shown in Section II, it can be solved simply by an ordinary LS estimation, i.e.,

$$\hat{\theta}_{LS} = \arg\min_{\theta} (\mathbf{A}\boldsymbol{\theta} - \mathbf{b})^T (\mathbf{A}\boldsymbol{\theta} - \mathbf{b}) = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}. \tag{7}$$

However, the ordinary LS solution in (7) is a coarse one, since it ignores the noise information as well as the correlation between source location and the auxiliary variable. The ordinary LS solution is often used as an initial solution for other improved localization algorithms. For the WLS method in [14]-[15], it requires the knowledge of parameters for the distribution function of TDOA measurement noise, i.e., the noise covariance matrix. To overcome these difficulties, we resort to CTLS to solve this problem.

A. Constrained Total Least-Squares Problem

Observing (6), both data matrix $\mathbf{A}$ and vector $\mathbf{b}$ contain the TDOA measurement noise and sensor position errors. Separating out the noise term in the measurement model, we have

$$\begin{aligned}\mathbf{A\theta}-\mathbf{b}&=\left(\mathbf{A}^0+\Delta\mathbf{A}\right)\mathbf{\theta}-\left(\mathbf{b}^0+\Delta\mathbf{b}\right)\\&=\Delta\mathbf{A\theta}-\Delta\mathbf{b},\end{aligned} \tag{8}$$

where

$$\Delta\mathbf{A}=\begin{bmatrix}\Delta s_{2x} & \Delta s_{2y} & \Delta r_{21}\\ \Delta s_{3x} & \Delta s_{3y} & \Delta r_{31}\\ \vdots & \vdots & \vdots\\ \Delta s_{Mx} & \Delta s_{My} & \Delta r_{M1}\end{bmatrix},$$

$$\Delta\mathbf{b}\approx\begin{bmatrix}\left(\mathbf{s}_2-\mathbf{s}_1^0\right)^T\Delta\mathbf{s}_2-r_{21}\Delta r_{21}\\ \left(\mathbf{s}_3-\mathbf{s}_1^0\right)^T\Delta\mathbf{s}_3-r_{31}\Delta r_{31}\\ \vdots\\ \left(\mathbf{s}_M-\mathbf{s}_1^0\right)^T\Delta\mathbf{s}_M-r_{M1}\Delta r_{M1}\end{bmatrix}.$$

The ith row of the error vector $\Delta\mathbf{b}$ can be obtained by omitting the quadratic noise term of the following formula

$$\Delta\mathbf{b}_i=\left(\mathbf{s}_i-\mathbf{s}_1^0\right)^T\Delta\mathbf{s}_i-r_{i1}\Delta r_{i1}-\frac{1}{2}\Delta\mathbf{s}_i^T\Delta\mathbf{s}_i+\frac{1}{2}\Delta r_{i1}^2.$$

Then, the matrix $\Delta\mathbf{A}$ and the vector $\Delta\mathbf{b}$ can be rewritten in the form related to noise term $\mathbf{n}_\mathbf{r}$ and $\mathbf{n}_\mathrm{S}$

$$\Delta\mathbf{A}=[\mathbf{G}_1\mathbf{n}_\mathrm{S}\quad\mathbf{G}_2\mathbf{n}_\mathrm{S}\quad\mathbf{G}_3\mathbf{n}_\mathbf{r}] \tag{9}$$

$$\Delta\mathbf{b}=\mathbf{h}_1\mathbf{n}_\mathrm{S}+\mathbf{h}_2\mathbf{n}_\mathbf{r} \tag{10}$$

where

$$\begin{aligned}\mathbf{n}_\mathbf{r}&=[\Delta r_{21}\quad\Delta r_{31}\quad\cdots\quad\Delta r_{M1}]^T\\ \mathbf{n}_\mathrm{S}&=[\Delta\mathbf{s}_2{}^T\quad\Delta\mathbf{s}_3{}^T\quad\cdots\quad\Delta\mathbf{s}_M{}^T]^T\end{aligned} \tag{11}$$

$$\begin{gathered}\mathbf{G}_1=diag\left(\mathbf{g}_{1,1},\mathbf{g}_{1,2},\ldots,\mathbf{g}_{1,M-1}\right),\ \mathbf{g}_{1,i}=\begin{bmatrix}1&0\end{bmatrix},\\ \mathbf{G}_2=diag\left(\mathbf{g}_{2,1},\mathbf{g}_{2,2},\ldots,\mathbf{g}_{2,M-1}\right),\ \mathbf{g}_{2,i}=\begin{bmatrix}0&1\end{bmatrix},\\ \mathbf{G}_3=\mathbf{I}_{M-1},\end{gathered} \tag{12}$$

$$\begin{gathered}\mathbf{h}_1=diag\left(\left(\mathbf{s}_2-\mathbf{s}_1\right)^T,\left(\mathbf{s}_3-\mathbf{s}_1\right)^T,\ldots,\left(\mathbf{s}_M-\mathbf{s}_1\right)^T\right),\\ \mathbf{h}_2=-diag\left(r_{21},r_{31},\ldots,r_{M1}\right).\end{gathered} \tag{13}$$

The residual term of the measurement model can be condensed into

$$\Delta\mathbf{A\theta}-\Delta\mathbf{b}=\mathbf{Fn}, \tag{14}$$

where

$$\mathbf{F}=\begin{bmatrix}\mathbf{F}_1&\mathbf{F}_2\end{bmatrix},\ \mathbf{n}=\begin{bmatrix}\mathbf{n}_\mathrm{S}^T&\mathbf{n}_r^T\end{bmatrix}^T$$

$$\mathbf{F}_1=v_1\mathbf{G}_1+v_2\mathbf{G}_2-\mathbf{h}_1,\ \mathbf{F}_2=r\mathbf{G}_3-\mathbf{h}_2,$$

$$\mathbf{v}\triangleq\begin{bmatrix}v_1&v_2\end{bmatrix}^T=\begin{bmatrix}x-s_{1x}&y-s_{1y}\end{bmatrix}^T.$$

Thus, we have the following pseudo-linear measurement model

$$\mathbf{A\theta}-\mathbf{b}=\mathbf{Fn}. \tag{15}$$

Consequently, the CTLS optimization problem can be constructed as

$$\begin{cases}\min\limits_{\theta}\|\mathbf{n}\|_2^2\\ \text{s.t. }\ \mathbf{A\theta}-\mathbf{b}=\mathbf{Fn},\\ \qquad\left(\mathbf{v}^T\mathbf{v}\right)^{1/2}-r=0.\end{cases} \tag{16}$$

where the second equational constraint considers the correlation between the source location and the auxiliary variables.

From (15), we have $\mathbf{n}=\mathbf{F}^\dagger\left(\mathbf{A\theta}-\mathbf{b}\right)$, where $\mathbf{F}^\dagger=\mathbf{F}^T\left(\mathbf{FF}^T\right)^{-1}$ is the right pseudoinverse of $\mathbf{F}$. Inserting the representation of $\mathbf{n}$ into the objective function in (16), we obtain

$$f\left(\theta\right)=\mathbf{n}^T\mathbf{n}=\left(\mathbf{A\theta}-\mathbf{b}\right)^T\left(\mathbf{FF}^T\right)^{-1}\left(\mathbf{A\theta}-\mathbf{b}\right). \tag{17}$$

The original optimization problem can be rewritten as

$$\begin{cases}\min\limits_{v_1,v_2,r}f\left(\mathbf{\theta}\right)\\ \text{s.t. }\ \left(\mathbf{v}^T\mathbf{v}\right)^{1/2}-r=0.\end{cases} \tag{18}$$

Assuming that there is an initial solution for $\mathbf{\theta}$, i.e., the terms $\left(\mathbf{FF}^T\right)^{-1}$ can be approximated, then the above optimization problem is a quadratic programming problem with equational constraints. For this optimization problem, the commonly used solvers are TSWLS in [14], and Newton's iterative algorithm. The TSWLS method requires knowledge of the distribution function and parameter information of the measurement noise and is not robust to large measurements noise. The first-order Newton iterative algorithm relies on the selection of initial values and has convergence problems. Second-order Newtonian iterative algorithms such as the interior point method [17] require strong conditions, which demand smoothness and convexity of the objective function and constraints. The use of the interior point method requires further relaxation and transformation of (18), leading to a degradation of the estimation performance. In this paper, we propose an ADMM-based CTLS algorithm.

B. Proposed CTLS-ADMM Algorithm for TDOA Localization with Sensor Position Errors

The ADMM algorithm provides a way to transform a multivariate optimization problem into a series of univariate optimization problem (i.e., alternating directions). The process of updating the parameters needs to be combined with a specific descent class of algorithms, such as gradient descent algorithms.

Reconstructing (18) in the form of an augmented Lagrangian function, we have

$$L(v_1,v_2,r,y)=(\mathbf{A}_1v_1+\mathbf{A}_2v_2+\mathbf{A}_3r-\mathbf{b})^T(\mathbf{F}\mathbf{F}^T)^{-1} \cdot(\mathbf{A}_1v_1+\mathbf{A}_2v_2+\mathbf{A}_3r-\mathbf{b})+y^T\left((\mathbf{v}^T\mathbf{v})^{1/2}-r\right) +\frac{\rho}{2}\left\|(\mathbf{v}^T\mathbf{v})^{1/2}-r\right\|_2^2, \tag{19}$$

where $\mathbf{A}\theta-\mathbf{b}=\mathbf{A}_1v_1+\mathbf{A}_2v_2+\mathbf{A}_3r-\mathbf{b}$. The 2-norm function at the end of the function is a penalty term, which can speed up the convergence of the algorithm and make the optimization problem smoother and more stable, where $\rho>0$ is the penalty term parameter.

For a simplified form, if we let $u=y/\rho$, then the augmented Lagrangian function can be written in the following scaled form

$$L(v_1,v_2,r,y)=(\mathbf{A}_1v_1+\mathbf{A}_2v_2+\mathbf{A}_3r-\mathbf{b})^T(\mathbf{F}\mathbf{F}^T)^{-1} \cdot(\mathbf{A}_1v_1+\mathbf{A}_2v_2+\mathbf{A}_3r-\mathbf{b}) +\frac{\rho}{2}\left\|(\mathbf{v}^T\mathbf{v})^{1/2}-r+u\right\|_2^2+\frac{\rho}{2}\|u\|_2^2. \tag{20}$$

The ADMM updates the primal variables (v_1,v_2,r) and the Lagrange multiplier u as follows

$$\begin{aligned} v_2^{(k+1)}&=\arg\min_{v_2}L\left(v_1^{(k+1)},v_2,r^{(k)},u^{(k)}\right),\\ r^{(k+1)}&=\arg\min_{r}L\left(v_1^{(k+1)},v_2^{(k+1)},r,u^{(k)}\right),\\ u^{(k+1)}&=u^{(k)}+\rho\left(\left(\mathbf{v}^{(k+1)T}\mathbf{v}^{(k+1)}\right)^{1/2}-r^{(k+1)}\right). \end{aligned} \tag{21}$$

where the initial solution $\left(v_1^{(0)},v_2^{(0)},r^{(0)},u^{(0)}\right)$ for the unknown parameters can be obtained using ordinary LS.

For v_1-updated, we perform a Taylor series expansion of function $L\left(v_1,v_2^{(k)},r^{(k)},u^{(k)}\right)$ at point $L\left(v_1^{(k)},v_2^{(k)},r^{(k)},u^{(k)}\right)$,

$$\begin{aligned} L\left(v_1,v_2^{(k)},r^{(k)},u^{(k)}\right)&\approx L\left(v_1^{(k)},v_2^{(k)},r^{(k)},u^{(k)}\right)\\ &+\left(v_1-v_1^{(k)}\right)^T\frac{\partial L\left(v_1^{(k)},v_2^{(k)},r^{(k)},u^{(k)}\right)}{\partial v_1}\\ &+\frac{1}{2}\left(v_1-v_1^{(k)}\right)^T\frac{\partial^2 L\left(v_1^{(k)},v_2^{(k)},r^{(k)},y^{(k)}\right)}{\partial {v_1}^2}\left(v_1-v_1^{(k)}\right). \end{aligned} \tag{22}$$

A necessary condition for the minimum of $L\left(v_1,v_2^{(k)},r^{(k)},u^{(k)}\right)$ is that the first-order derivative is zero $\partial L\left(v_1,v_2^{(k)},r^{(k)},u^{(k)}\right)/\partial v_1=0$. It follows from (22) that,

$$\begin{aligned} \frac{\partial L\left(v_1,v_2^{(k)},r^{(k)},u^{(k)}\right)}{\partial v_1}&=\frac{\partial L\left(v_1^{(k)},v_2^{(k)},r^{(k)},u^{(k)}\right)}{\partial v_1}\\ &+\frac{\partial^2 L\left(v_1^{(k)},v_2^{(k)},r^{(k)},u^{(k)}\right)}{\partial {v_1}^2}\left(v_1-v_1^{(k)}\right)=0. \end{aligned} \tag{23}$$

Then, the update process for v_1 is

$$v_1^{(k+1)}=v_1^{(k)}-\left(\frac{\partial^2 L}{\partial {v_1}^2}\right)^{-1}\cdot\left.\frac{\partial L}{\partial v_1}\right|_{\left(v_1^{(k)},v_2^{(k)},r^{(k)},u^{(k)}\right)}. \tag{24}$$

Taking the derivative of (20) with respect to v_1, we have

$$\frac{\partial L}{\partial v_1}=\frac{\partial f(\boldsymbol{\theta})}{\partial v_1}+\rho\left((\mathbf{v}^T\mathbf{v})^{1/2}-r+u\right)\left((\mathbf{v}^T\mathbf{v})^{-1/2}v_1\right), \tag{25}$$

$$\frac{\partial f(\boldsymbol{\theta})}{\partial v_1}=2\mathbf{A}_1^T(\mathbf{F}\mathbf{F}^T)^{-1}(\mathbf{A}\boldsymbol{\theta}-\mathbf{b})+(\mathbf{A}\boldsymbol{\theta}-\mathbf{b})^T(-\Psi \mathrm{P}_1\Psi)(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}), \tag{26}$$

where

$$\Psi=(\mathbf{F}\mathbf{F}^T)^{-1},\frac{\partial\Psi}{\partial v_1}=-\Psi\mathrm{P}_1\Psi,$$

$$\mathrm{P}_1=\frac{\partial(\mathbf{F}\mathbf{F}^T)}{\partial v_1}=\frac{\partial(\mathbf{F}_1\mathbf{F}_1^T)}{\partial v_1}=\mathbf{G}_1\mathbf{F}_1^T+\mathbf{F}_1\mathbf{G}_1^T.$$

Taking the derivative of $\partial L/\partial v_1$ with respect to v_1, the derivative is simplified as follows

$$\begin{aligned} \frac{\partial^2 L}{\partial {v_1}^2}&=\frac{\partial^2 f(\boldsymbol{\theta})}{\partial {v_1}^2}+\rho(\mathbf{v}^T\mathbf{v})^{-1}{v_1}^2+\\ &\rho\left((\mathbf{v}^T\mathbf{v})^{1/2}-r+u\right)\cdot\left(-(\mathbf{v}^T\mathbf{v})^{-3/2}{v_1}^2+(\mathbf{v}^T\mathbf{v})^{-1/2}\right), \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{\partial^2 f(\theta)}{\partial v_1^2}&=2\mathbf{A}_1^T(\mathbf{F}\mathbf{F}^T)^{-1}\mathbf{A}_1+4\mathbf{A}_1^T(-\Psi\mathrm{P}_1\Psi)(\mathbf{A}\theta-\mathbf{b})\\ &+(\mathbf{A}\theta-\mathbf{b})^T\frac{\partial(-\Psi\mathrm{P}_1\Psi)}{\partial v_1}(\mathbf{A}\theta-\mathbf{b}), \end{aligned} \tag{28}$$

where

$$\frac{\partial(-\Psi\mathrm{P}_1\Psi)}{\partial v_1}=(\Psi\mathrm{P}_1\Psi)\mathrm{P}_1\Psi-2\Psi(\mathbf{G}_1\mathbf{G}_1^T)\Psi+\Psi\mathrm{P}_1(\Psi\mathrm{P}_1\Psi). \tag{29}$$

By adopting the similar approach, we can obtain the results of v_2-updates and r-updated as

$$v_2^{(k+1)}=v_2^{(k)}-\left(\frac{\partial^2 L}{\partial {v_2}^2}\right)^{-1}\cdot\left.\frac{\partial L}{\partial v_2}\right|_{\left(v_1^{(k+1)},v_2^{(k)},r^{(k)},u^{(k)}\right)} \tag{30}$$

$$r^{(k+1)}=r^{(k)}-\left(\frac{\partial^2 L}{\partial r^2}\right)^{-1}\cdot\left.\frac{\partial L}{\partial r}\right|_{\left(v_1^{(k+1)},v_2^{(k+1)},r^{(k)},u^{(k)}\right)} \tag{31}$$

For specific derivative calculations for v_2 and r see the Appendix. In summary, the ADMM algorithm steps are illustrated as **Algorithm 1.**

Algorithm 1: Proposed ADMM-CTLS Algorithm for TDOA Localization with Sensor Position Errors

1. **Initialization**: Set the iteration $k=0$, $k_{\max}=1000$. Obtain the coarse solution $\left(v_1^{(0)}, v_2^{(0)}, r^{(0)}, u^{(0)}\right)$ using LS.
2. **For** $k=0$ to $k_{\max}$
   Obtain the matrix **F** using the estimate $\left(v_1^{(k)}, v_2^{(k)}, r^{(k)}, u^{(k)}\right)$ from the kth iteration.
3. v_1 **-update**: Using (25)-(29) to compute the first-order and second-order derivatives with respect to v_1 and substitute them into the following equation to obtain $v_1^{(k+1)}$.
   $$v_1^{(k+1)} = v_1^{(k)} - \left(\frac{\partial^2 L}{\partial v_1^2}\right)^{-1} \cdot \left.\frac{\partial L}{\partial v_1}\right|_{\left(v_1^{(k)}, v_2^{(k)}, r^{(k)}, u^{(k)}\right)}$$
   v_2 **-update** is the same as v_1 -update,
   $$v_2^{(k+1)} = v_2^{(k)} - \left(\frac{\partial^2 L}{\partial v_2^2}\right)^{-1} \cdot \left.\frac{\partial L}{\partial v_2}\right|_{\left(v_1^{(k+1)}, v_2^{(k)}, r^{(k)}, u^{(k)}\right)}$$
   r **-update** is the same as the v_1 -update,
   $$r^{(k+1)} = r^{(k)} - \left(\frac{\partial^2 L}{\partial r^2}\right)^{-1} \cdot \left.\frac{\partial L}{\partial r}\right|_{\left(v_1^{(k+1)}, v_2^{(k+1)}, r^{(k)}, u^{(k)}\right)}$$
   u **-update:**
   $$u^{(k+1)} = u^{(k)} + \rho\left(\left(\mathbf{v}^{(k+1)T}\mathbf{v}^{(k+1)}\right)^{1/2} - r^{(k+1)}\right).$$
   Terminate when $\left\|\mathbf{v}^{(k+1)} - \mathbf{v}^{(k)}\right\|_2 \le \delta.$
4. **End** $\hat{\mathbf{x}} = \mathbf{v}^{(k+1)} + \mathbf{s}_1$.

IV. Simulation Results

To verify the effectiveness of the proposed algorithm, simulations are performed in this section. We apply the proposed algorithm to two localization scenarios in terms of different types of errors. The results are compared with the two-step closed form WLS method considering SPE in [25], the constrained TLS method in [29] and the improved two-step constrained TLS method in [31]. Symbols used for the simulations are as follows: 1) 'TSWLS-S' denotes the two-step weighted least squares algorithm considering sensor position errors in [25]. 2) 'CTLS' denotes the constrained total least squares method in [29]. 3) 'TSCTLS' denotes the improved two-step constrained total least squares algorithm in [31]. 4) 'Proposed method' denotes the proposed CTLS-ADMM described in section III.

Note that the TSWLS-S method takes into account the SPE that might affect source localization accuracy, whereas the CTLS and TSCTLS methods ignore the effect of SPE. Thus, the TSWLS-S method and the proposed method are mainly used for performance comparisons.

A. Simulation Settings.

The scenario contains $M=5$ sensors, and their nominal positions are given in Table I. The coordinates of the source are [1200, 800] m. 1000 Monte Carlo simulations are performed for each test. The localization accuracy is evaluated in terms of the root mean square error (RMSE) and the bias norm of the source position, which are defined as

TABLE I. Position of the Sensors

Sensors	s_{ix} (m)	s_{iy} (m)
1	0	0
2	160	420
3	340	520
4	580	300
5	780	180

$$RMSE = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left\|\hat{\mathbf{x}}_i - \mathbf{x}\right\|_2^2} \tag{32}$$

$$Bias = \left\|\frac{1}{m}\sum_{i=1}^{m}\hat{\mathbf{x}}_i - \mathbf{x}\right\|_2 \tag{33}$$

where $\|\cdot\|_2$ denotes the 2-norm, m is the number of Monte Carlo runs, $\hat{\mathbf{x}}$ is the position estimate of the source in the ith test.

For the simulation, both the TDOA measurement error and the sensor position error are sampled from zero mean Gaussian white noise. The penalty term coefficient of the ADMM algorithm is set to $\rho=0.1$ and the termination condition parameter is set to $\delta=10^{-3}$.

B. The Impact of the Measurement Noise

In the scenario of this subsection, we compare the localization performance of four methods in terms of different levels of measurement noise. The standard deviation of TDOA measurements is taken from 0 ns to 10 ns. The standard deviation of sensor position errors is set relatively small and fixed at 1m.

Comparison of the RMSE and bias norm with varying measurement noise is shown in Figs. 1 and 2. From Fig. 1, it can be observed that the RMSE of all algorithms becomes larger with the increase in TDOA measurement noise, whereas the TSWLS-S and the proposed method are clearly superior to the other two methods. This conclusion can be justified because the construction process of these two algorithms takes into account the effect of sensor position errors on the localization performance. Moreover, the proposed algorithm almost reaches the corresponding CRLB. When the TDOA noise gradually increases, the proposed algorithm haves more robust than the comparative TSWLS-S.

From Fig. 2, it is obviously seen that when the TDOA measurement noise is small enough, the bias norm of all algorithms remains low. However, when the TDOA measurement noise is larger than 7 ns, the bias norm of the CTLS and TSCTLS algorithms experiences a sharp increase. In general, the performance of the proposed method is superior to that of TSWLS-S and significantly outperforms both the CTLS and TSCTLS methods.

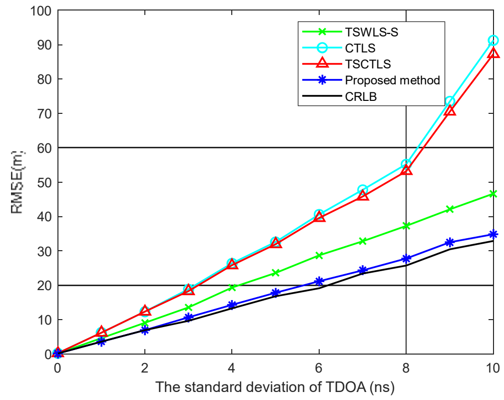

Fig. 1 RMSE versus TDOA noise

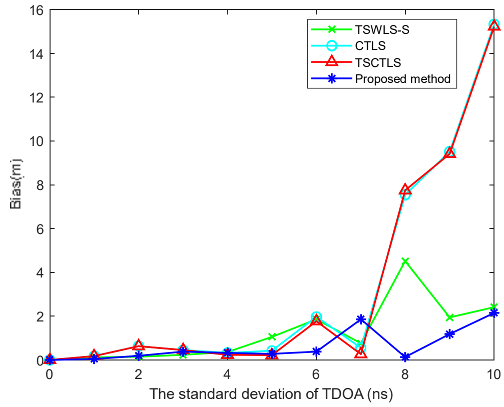

Fig. 2 Bias norm versus TDOA noise

C. The Impact of the Sensor Position error

The above results show that the TSWLS-S algorithm and proposed method perform well since they consider the sensor position error. In this subsection, we focus on conduct performance comparisons between them when changing the level of sensor location errors. In this scenario, the standard deviation of the TDOA measurement noise is set to a medium size of 5ns. The standard deviation of the sensor position error increases gradually from 0 m to 10 m.

Comparisons of the RMSE and bias norm with varying SPE are shown in Fig. 3-4. We can see from Fig. 3 that the proposed method outperforms the TSWLS-S algorithm at any value of SPE with smaller RMSE. Fig.3 also shows that the proposed method can approach CRLB better. Fig. 4 indicates that the bias norm of the two methods is relatively close to each other when the SPE is less than 7m. However, the bias norm of the comparative TSWLS-S increase faster than the proposed method as the SPE gets large.

V. Conclusion

To address the problem of source localization with sensor position errors and large measurement errors, the CTLS-ADMM algorithm using TDOA measurements is presented in this paper. Simulation results show that the proposed algorithm is superior to the existing ones, with smaller RMSE and bias norms.

Future work includes developing robust localization methods in the presence of both TDOA measurement outliers and sensor position errors.

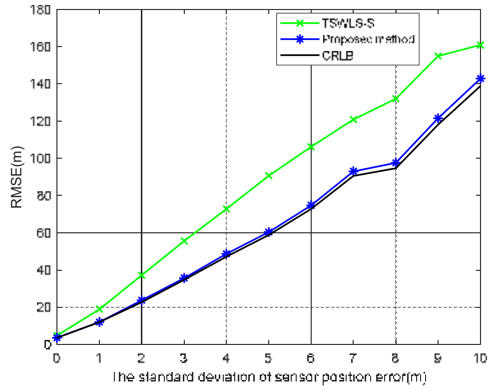

Fig. 3 RMSE versus sensor position errors

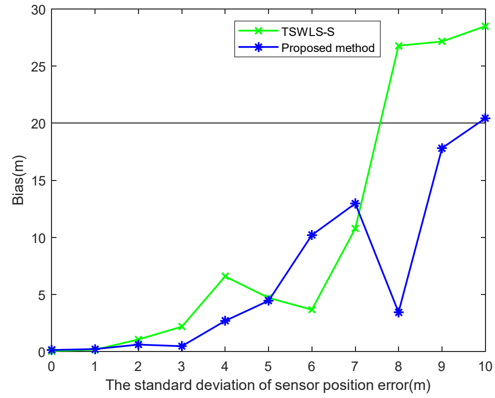

Fig. 4 Bias norm versus sensor position errors

Appendix

A. v_2-update

Taking the derivative of (20) with respect to v_2, we have

$$\frac{\partial L}{\partial v_2} = \frac{\partial f(\boldsymbol{\theta})}{\partial v_2} + \rho\left(\left(\mathbf{v}^T\mathbf{v}\right)^{1/2} - r + u\right)\left(\left(\mathbf{v}^T\mathbf{v}\right)^{-1/2} v_2\right), \quad (34)$$

$$\frac{\partial f(\boldsymbol{\theta})}{\partial v_2} = 2\mathbf{A}_2^T\left(\mathbf{F}\mathbf{F}^T\right)^{-1}\left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right) + \left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right)^T\left(-\Psi \mathrm{P}_2 \Psi\right)\left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right), \quad (35)$$

where

$$\frac{\partial \Psi}{\partial v_2} = -\Psi \mathrm{P}_2 \Psi,$$

$$\mathrm{P}_2 = \frac{\partial\left(\mathbf{F}\mathbf{F}^T\right)}{\partial v_2} = \frac{\partial\left(\mathbf{F}_1\mathbf{F}_1^T\right)}{\partial v_2} = \mathbf{G}_2\mathbf{F}_2^{\,T} + \mathbf{F}_2\mathbf{G}_2^{\,T}.$$

Taking the derivative of $\partial L/\partial v_2$ with respect to v_2, the derivative is simplified as follows

$$\begin{aligned}\frac{\partial^2 L}{\partial v_2^2} &= \frac{\partial^2 f(\boldsymbol{\theta})}{\partial v_2^2} + \rho\left(\mathbf{v}^T\mathbf{v}\right)^{-1} v_2^{\,2} + \\ &\rho\left(\left(\mathbf{v}^T\mathbf{v}\right)^{1/2} - r + u\right)\cdot\left(-\left(\mathbf{v}^T\mathbf{v}\right)^{-3/2} v_2^{\,2} + \left(\mathbf{v}^T\mathbf{v}\right)^{-1/2}\right),\end{aligned} \quad (36)$$

$$\frac{\partial^2 f(\boldsymbol{\theta})}{\partial v_2^2} = 2\mathbf{A}_2^T\left(\mathbf{F}\mathbf{F}^T\right)^{-1}\mathbf{A}_2 + 4\mathbf{A}_2^T\left(-\Psi \mathrm{P}_2 \Psi\right)\left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right) + \left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right)^T \frac{\partial\left(-\Psi \mathrm{P}_2 \Psi\right)}{\partial v_2}\left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right), \tag{37}$$

where

$$\frac{\partial\left(-\Psi \mathrm{P}_2 \Psi\right)}{\partial v_2} = \left(\Psi \mathrm{P}_2 \Psi\right)\mathrm{P}_2\Psi - 2\Psi\left(\mathbf{G}_2\mathbf{G}_2^T\right)\Psi + \Psi \mathrm{P}_2\left(\Psi \mathrm{P}_2 \Psi\right).$$

B. r *-update*

Taking the derivative of (20) with respect to r, we have

$$\frac{\partial L}{\partial r} = \frac{\partial f(\boldsymbol{\theta})}{\partial r} - \rho\left(\left(\mathbf{v}^T\mathbf{v}\right)^{1/2} - r + u\right), \tag{38}$$

$$\frac{\partial f(\boldsymbol{\theta})}{\partial r} = 2\mathbf{A}_3^T\left(\mathbf{F}\mathbf{F}^T\right)^{-1}\left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right) + \left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right)^T\left(-\Psi \mathrm{P}_3 \Psi\right)\left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right). \tag{39}$$

where

$$\frac{\partial \Psi}{\partial r} = -\Psi \mathrm{P}_3 \Psi,$$

$$\mathrm{P}_3 = \frac{\partial\left(\mathbf{F}\mathbf{F}^T\right)}{\partial r} = \frac{\partial\left(\mathbf{F}_2\mathbf{F}_2^{\;T}\right)}{\partial v_1} = \mathbf{G}_3\mathbf{F}_2^{\;T} + \mathbf{F}_2\mathbf{G}_3^{\;T}.$$

Taking the derivative of $\partial L/\partial r$ with respect to r, the derivative is simplified as follows

$$\frac{\partial^2 L}{\partial r^2} = \frac{\partial^2 f(\boldsymbol{\theta})}{\partial r^2} + \rho \tag{40}$$

$$\frac{\partial^2 f(\boldsymbol{\theta})}{\partial r^2} = 2\mathbf{A}_3^T\left(\mathbf{F}\mathbf{F}^T\right)^{-1}\mathbf{A}_3 + 4\mathbf{A}_3^T\left(-\Psi \mathrm{P}_3 \Psi\right)\left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right) + \left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right)^T \frac{\partial\left(-\Psi \mathrm{P}_3 \Psi\right)}{\partial r}\left(\mathbf{A}\boldsymbol{\theta}-\mathbf{b}\right). \tag{41}$$

where

$$\frac{\partial\left(-\Psi \mathrm{P}_3 \Psi\right)}{\partial r} = \left(\Psi \mathrm{P}_3 \Psi\right)\mathrm{P}_3\Psi - 2\Psi\left(\mathbf{G}_3\mathbf{G}_3^T\right)\Psi + \Psi \mathrm{P}_3\left(\Psi \mathrm{P}_3 \Psi\right).$$

REFERENCES

[1] Y. Wang, W. Dong, G. -C. Sun, Z. Zhang, M. Xing and X. Yang, "A CLEAN-Based Synthetic Aperture Passive Localization Algorithm for Multiple Signal Sources," IEEE Journal on Miniaturization for Air and Space Systems, vol. 3, no. 4, pp. 294-301, Dec. 2022.

[2] B. Jin, X. Xu, Y. Zhu, T. Zhang, and Q. Fei, "Single-source aided semiautonomous passive location for correcting the position of an underwater vehicle," IEEE Sensors J., vol. 19, no. 9, pp. 3267–3275, May 2019.

[3] M. Seifeldin, A. Saeed, A. E. Kosba, A. El-Keyi and M. Youssef, "Nuzzer: A Large-Scale Device-Free Passive Localization System for Wireless Environments," IEEE Transactions on Mobile Computing, vol. 12, no. 7, pp. 1321–1334, 2013.

[4] Y. Liang and H. Li, "Source Localization with Spatially Distributed Active and Passive Sensors," 2024 58th Annual Conference on Information Sciences and Systems (CISS), Princeton, NJ, USA, 2024, pp. 1-6

[5] S. Gao, S. Zhang, G. Wang, and Y. Li, "Robust second-order cone relaxation for TW-TOA-based localization with clock imperfection," IEEE Signal Process. Lett., vol. 23, no. 8, pp. 1047–1051, Aug. 2016.

[6] X. Qu and L. Xie, "An efficient convex constrained weighted least squares source localization algorithm based on TDOA measurements," Signal Process., vol. 119, pp. 142–152, Feb. 2016.

[7] X. Qu, L. Xie, and W. Tan, "Iterative constrained weighted least squares source localization using TDOA and FDOA measurements," IEEE Trans. Signal Process., vol. 65, no. 15, pp. 3990–4003, Aug. 2017.

[8] S. Tomic, M. Beko, and R. Dinis, "RSS-based localization in wireless sensor networks using convex relaxation: Noncooperative and cooperative schemes," IEEE Trans. Veh. Technol., vol. 64, no. 5, pp. 2037–2050, May 2015.

[9] L. Cong and W. Zhuang, "Hybrid TDOA/AOA mobile user location for wideband CDMA cellular systems," IEEE Trans. Wireless Commun, vol. 1, pp. 439–447, Jul. 2002.

[10] M. R. Gholami, S. Gezici, and E. G. Strom, "TDOA based positioning in the presence of unknown clock skew," IEEE Trans. Commun., vol. 61, no. 6, pp. 2522–2534, Jun. 2013.

[11] W. H. Foy, "Position-location solution by Taylor-series estimation," IEEE Trans. Aerosp. Elect. Syst., vol. AES-12, pp. 187–194, Mar. 1976.

[12] K. W. K. Lui, F. K. W. Chan and H. C. So, "Semidefinite Programming Approach for Range-Difference Based Source Localization," in IEEE Transactions on Signal Processing, vol. 57, no. 4, pp. 1630-1633, April 2009.

[13] J. O. Smith and J. S. Abel, "Closed-form least-squares source location estimation from range-difference measurements," IEEE Trans. Acoust., Speech, Signal Process., vol. ASSP-35, no. 12, pp. 1661–1669, Dec. 1987.

[14] Y. T. Chan and K. C. Ho, "A simple and efficient estimator for hyperbolic location," IEEE Trans. Signal Processing, vol. 42, pp. 1905–1915, Aug. 1994.

[15] L. Lin, H. C. So, F. K. W. Chan, Y. T. Chan, and K. C. Ho, "A new constrained weighted least squares algorithm for TDOA-based localization," Signal Process., vol. 93, no. 11, pp. 2872–2878, 2013.

[16] C. Ho, "Bias reduction for an explicit solution of source localization using TDOA," IEEE Trans. Signal Process., vol. 60, no. 5, pp. 2101–2114, May 2012.

[17] G. Wang, A. M.-C. So, and Y. Li, "Robust convex approximation methods for TDOA-based localization under NLOS conditions," IEEE Trans. Signal Process., vol. 64, no. 13, pp. 3281–3296, Jul. 2016.

[18] M. R. Gholami, S. Gezici, and E. G. Strom, "A concave-convex procedure for TDOA based positioning," IEEE Commun. Lett., vol. 17, no. 4, pp. 765–768, Apr. 2013.

[19] B. Huang, L. Xie, and Z. Yang, "TDOA-based source localization with distance-dependent noises," IEEE Trans. Wireless Commun., vol. 14, no. 1, pp. 468–480, Jan. 2015.

[20] N. Patwari, J. N. Ash, S. Kyperountas, A. O. Hero, R. L. Moses, and N. S. Correal, "Locating the nodes: cooperative localization in wireless sensor networks," IEEE Signal Process. Mag., vol. 22, no. 4, pp. 54–69, July 2005.

[21] G. Mao, B. Fidan, and B. D. O. Anderson, "Wireless sensor network localization techniques," Comput. Netw., vol. 51, no. 10, pp. 2529–2553, Jul. 2007.

[22] K. Yu, J. P. Montillet, A. Rabbachin, P. Cheong, and I. Oppermann, "UWB location and tracking for wireless embedded networks," Elsevier Signal Process., vol. 86, no. 9, pp. 2153–2171, Sept. 2006.

[23] K. W. K. Lui, W. -K. Ma, H. C. So and F. K. W. Chan, "Semi-Definite Programming Algorithms for Sensor Network Node Localization With Uncertainties in Anchor Positions and/or Propagation Speed," IEEE Transactions on Signal Processing, vol. 57, no. 2, pp. 752-763, Feb. 2009.

[24] Y. Rockah and P. M. Schultheiss, "Array shape calibration using sources in unknown location, Part I: Far-field source," IEEE Trans. Acoust., Speech, Signal Process., vol. ASSP-35, pp. 286–299, Mar. 1987.

[25] K. C. Ho, L. Kovavisaruch, and H. Parikh, "Source localization using TDOA with erroneous receiver positions," in Proc. IEEE Acoust., Speech, Signal Process., May 2004, vol. 3, pp. 453–456.

[26] K. Ho,d X. Lu, L. Kovavisaruch, Source localization using TDOA and FDOA measurements in the presence of receiver location errors: analysis and solution, IEEE Trans. Signal Process. 55(2) 684–696.

[27] G. Golub. Some modified matrix eigenvalue problems. SIAM Review, 15:318–344, 1973.

[28] G. Golub and C. Van Loan. An analysis of the total least squares problem. SIAM J. Numer. Anal., 17:883–893, 1980.

[29] K. Yang, J. An, X. Bu and G. Sun, "Constrained Total Least-Squares Location Algorithm Using Time-Difference-of-Arrival Measurements," in IEEE Transactions on Vehicular Technology, vol. 59, no. 3, pp. 1558-1562, March 2010.

[30] Doganc¸ay, Kutluyil. ”Bearings-only target localization using total least squares.” Signal Processing 85.9(2005):1695-1710.

[31] Q. Li, B. Chen and M. Yang, "Improved Two-Step Constrained Total Least-Squares TDOA Localization Algorithm Based on the Alternating Direction Method of Multipliers," in IEEE Sensors Journal, vol. 20, no. 22, pp. 13666-13673, 15 Nov.15, 2020.

[32] S. Boyd, “Distributed optimization and statistical learning via the alternating direction method of multipliers,” Found. Trends Mach. Learn., vol. 3, no. 1, pp. 1–122, 2010.

[33] J. Nocedal and S. J. Wright, Numerical Optimization. NewYork, NY, USA: Springer-Verlag, 1999.

[34] J. Eckstein and D. P. Bertsekas, “On the Douglas—Rachford splitting method and the proximal point algorithm for maximal monotone operators,” Math. Program., vol. 55, nos. 1–3, pp. 293–318, Apr. 1992.

[35] T. Jia, H. Wang, X. Shen, J. Gao and X. Liu, "An accurate TDOA-AOA localization method using structured total least squares," OCEANS 2017 - Aberdeen, 2017, pp. 1-6.